

# RIGIDITY THEOREMS OF $\lambda$ -HYPERSURFACES

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**ABSTRACT.** Since  $n$ -dimensional  $\lambda$ -hypersurfaces in the Euclidean space  $\mathbb{R}^{n+1}$  are critical points of the weighted area functional for the weighted volume-preserving variations, in this paper, we study the rigidity properties of complete  $\lambda$ -hypersurfaces. We give a gap theorem of complete  $\lambda$ -hypersurfaces with polynomial area growth. By making use of the generalized maximum principle for  $\mathcal{L}$  of  $\lambda$ -hypersurfaces, we prove a rigidity theorem of complete  $\lambda$ -hypersurfaces.

## 1. INTRODUCTION

Let  $X : M \rightarrow \mathbb{R}^{n+1}$  be a smooth  $n$ -dimensional immersed hypersurface in the  $(n+1)$ -dimensional Euclidean space  $\mathbb{R}^{n+1}$ . In [4], Cheng and Wei have introduced notation of the weighted volume-preserving mean curvature flow, which is defined as the following: a family  $X(\cdot, t)$  of smooth immersions

$$X(\cdot, t) : M \rightarrow \mathbb{R}^{n+1}$$

with  $X(\cdot, 0) = X(\cdot)$  is called a *weighted volume-preserving mean curvature flow* if

$$(1.1) \quad \frac{\partial X(t)}{\partial t} = -\alpha(t)N(t) + \mathbf{H}(t)$$

holds, where

$$\alpha(t) = \frac{\int_M H(t) \langle N(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu},$$

$\mathbf{H}(t) = \mathbf{H}(\cdot, t)$  and  $N(t)$  denote the mean curvature vector and the normal vector of hypersurface  $M_t = X(M^n, t)$  at point  $X(\cdot, t)$ , respectively and  $N$  is the unit normal vector of  $X : M \rightarrow \mathbb{R}^{n+1}$ . One can prove that the flow (1.1) preserves the weighted volume  $V(t)$  defined by

$$V(t) = \int_M \langle X(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu.$$

The weighted area functional  $A : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$  is defined by

$$A(t) = \int_M e^{-\frac{|X(t)|^2}{2}} d\mu_t,$$

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where  $d\mu_t$  is the area element of  $M$  in the metric induced by  $X(t)$ . Let  $X(t) : M \rightarrow \mathbb{R}^{n+1}$  with  $X(0) = X$  be a variation of  $X$ . If  $V(t)$  is constant for any  $t$ , we call  $X(t) : M \rightarrow \mathbb{R}^{n+1}$  a *weighted volume-preserving variation* of  $X$ . Cheng and Wei [4] have proved that  $X : M \rightarrow \mathbb{R}^{n+1}$  is a critical point of the weighted area functional  $A(t)$  for all weighted volume-preserving variations if and only if there exists constant  $\lambda$  such that

$$(1.2) \quad \langle X, N \rangle + H = \lambda.$$

An immersed hypersurface  $X(t) : M \rightarrow \mathbb{R}^{n+1}$  is called a  $\lambda$ -hypersurface if the equation (1.2) is satisfied.

**Remark 1.1.** If  $\lambda = 0$ , then the  $\lambda$ -hypersurface is a self-shrinker of the mean curvature flow. Hence, the  $\lambda$ -hypersurface is a generalization of the self-shrinker.

**Example 1.1.** The  $n$ -dimensional sphere  $S^n(r)$  with radius  $r > 0$  is a compact  $\lambda$ -hypersurface in  $\mathbb{R}^{n+1}$  with  $\lambda = \frac{n}{r} - r$ .

**Example 1.2.** For  $1 \leq k \leq n-1$ , the  $n$ -dimensional cylinder  $S^k(r) \times \mathbb{R}^{n-k}$  with radius  $r > 0$  is a complete and non-compact  $\lambda$ -hypersurface in  $\mathbb{R}^{n+1}$  with  $\lambda = \frac{k}{r} - r$ .

**Example 1.3.** The  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is a complete and non-compact  $\lambda$ -hypersurface in  $\mathbb{R}^{n+1}$  with  $\lambda = 0$ .

**Definition 1.1.** If  $X : M \rightarrow \mathbb{R}^{n+1}$  is an  $n$ -dimensional hypersurface in  $\mathbb{R}^{n+1}$ , we say that  $M$  has polynomial area growth if there exist constant  $C$  and  $d$  such that for all  $r \geq 1$ ,

$$(1.3) \quad \text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0) \cap X(M)} d\mu \leq Cr^d,$$

where  $B_r(0)$  is a standard ball in  $\mathbb{R}^{n+1}$  with radius  $r$  and centered at the origin.

In [4], Cheng and Wei have studied properties of complete  $\lambda$ -hypersurfaces with polynomial area growth. They have proved that a complete and non-compact  $\lambda$ -hypersurface  $X : M \rightarrow \mathbb{R}^{n+1}$  in the Euclidean space  $\mathbb{R}^{n+1}$  has polynomial area growth if and only if  $X : M \rightarrow \mathbb{R}^{n+1}$  is a complete proper hypersurface. Furthermore, there is a positive constant  $C$  such that for  $r \geq 1$ ,

$$(1.4) \quad \text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0) \cap X(M)} d\mu \leq Cr^{n + \frac{\lambda^2}{2} - 2\beta - \frac{\inf H^2}{2}},$$

where  $\beta = \frac{1}{4} \inf(\lambda - H)^2$ .

In this paper, we study the rigidity of complete  $\lambda$ -hypersurfaces. We will prove the following:

**Theorem 1.1.** Let  $X : M \rightarrow \mathbb{R}^{n+1}$  be an  $n$ -dimensional complete  $\lambda$ -hypersurface with polynomial area growth in the Euclidean space  $\mathbb{R}^{n+1}$ . Then  $X : M \rightarrow \mathbb{R}^{n+1}$  satisfies one of the following:

- (1)  $X : M \rightarrow \mathbb{R}^{n+1}$  is isometric to the sphere  $S^n(r)$  with radius  $r > 0$ ,
- (2)  $X : M \rightarrow \mathbb{R}^{n+1}$  is isometric to the Euclidean space  $\mathbb{R}^n$ ,
- (3)  $X : M \rightarrow \mathbb{R}^{n+1}$  is isometric to the cylinder  $S^1(r) \times \mathbb{R}^{n-1}$ ,

- (4)  $X : M \rightarrow \mathbb{R}^{n+1}$  is isometric to the cylinder  $S^{n-1}(r) \times \mathbb{R}$ ,
- (5)  $X : M \rightarrow \mathbb{R}^{n+1}$  is isometric to the cylinder  $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$  for  $2 \leq k \leq n-2$ ,
- (6) there exists  $p \in M$  such that the squared norm  $S$  of the second fundamental form and the mean curvature  $H$  of  $X : M \rightarrow \mathbb{R}^{n+1}$  satisfy

$$(1.5) \quad \left( \sqrt{S(p) - \frac{H^2(p)}{n}} + |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n}(H(p) - \lambda)^2 > 1 + \frac{n\lambda^2}{4(n-1)}.$$

**Corollary 1.1.** *Let  $X : M \rightarrow \mathbb{R}^{n+1}$  be an  $n$ -dimensional complete  $\lambda$ -hypersurface with polynomial area growth in the Euclidean space  $\mathbb{R}^{n+1}$ . If the squared norm  $S$  of the second fundamental form and the mean curvature  $H$  of  $X : M \rightarrow \mathbb{R}^{n+1}$  satisfies*

$$(1.6) \quad \left( \sqrt{S - \frac{H^2}{n}} + |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n}(H - \lambda)^2 \leq 1 + \frac{n\lambda^2}{4(n-1)},$$

*then  $X : M \rightarrow \mathbb{R}^{n+1}$  is isometric to one of the following:*

- (1) the sphere  $S^n(r)$  with radius  $0 < r \leq \sqrt{n}$ ,
- (2) the Euclidean space  $\mathbb{R}^n$ ,
- (3) the cylinder  $S^1(r) \times \mathbb{R}^{n-1}$  with radius  $r > 0$  and  $n = 2$  or with radius  $r \geq 1$  and  $n > 2$ ,
- (4) the cylinder  $S^{n-1}(r) \times \mathbb{R}$  with radius  $r > 0$  and  $n = 2$  or with radius  $r \geq \sqrt{n-1}$  and  $n > 2$ ,
- (5) the cylinder  $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$  for  $2 \leq k \leq n-2$ .

**Remark 1.2.** *If  $\lambda = 0$ , that is,  $X : M \rightarrow \mathbb{R}^{n+1}$  is an  $n$ -dimensional complete self-shrinker, our condition (1.6) becomes  $S \leq 1$ . Hence, our theorem is a general generalization of Cao and Li [1] and Le and Sesum [11] to  $\lambda$ -hypersurfaces. On study of complete self-shrinkers, see [2], [3], [5], [6], [7, 8], [9, 10].*

**Theorem 1.2.** *Let  $X : M \rightarrow \mathbb{R}^{n+1}$  be an  $n$ -dimensional complete  $\lambda$ -hypersurface with polynomial area growth in the Euclidean space  $\mathbb{R}^{n+1}$ . If*

$$(1.7) \quad \left( H - \frac{\lambda}{2} \right)^2 \geq n + \frac{\lambda^2}{4},$$

*then  $(H - \frac{\lambda}{2})^2 \equiv n + \frac{\lambda^2}{4}$  and  $M$  is isometric to the sphere  $S^n(r)$  with radius  $r > 0$ .*

If we do not assume that  $X : M \rightarrow \mathbb{R}^{n+1}$  has polynomial area growth, we can prove the following:

**Theorem 1.3.** *Let  $X : M \rightarrow \mathbb{R}^{n+1}$  be an  $n$ -dimensional complete  $\lambda$ -hypersurface in the Euclidean space  $\mathbb{R}^{n+1}$ . If the squared norm  $S$  of the second fundamental form and the mean curvature  $H$  of  $X : M \rightarrow \mathbb{R}^{n+1}$  satisfy*

$$(1.8) \quad \sup \left\{ \left( \sqrt{S - \frac{H^2}{n}} + |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n}(H - \lambda)^2 \right\} < 1 + \frac{n\lambda^2}{4(n-1)},$$

*then  $X : M \rightarrow \mathbb{R}^{n+1}$  is isometric to one of the following:*

- (1) the sphere  $S^n(r)$  with radius  $r < \sqrt{n}$ ,
- (2) the Euclidean space  $\mathbb{R}^n$ .

**Remark 1.3.** If  $\lambda = 0$ , that is,  $X : M \rightarrow \mathbb{R}^{n+1}$  is an  $n$ -dimensional complete self-shrinker, our condition (1.8) becomes  $\sup S < 1$ . Our theorem is a general generalization of Cheng and Peng [2] to  $\lambda$ -hypersurfaces.

We next give the following:

**Proposition 1.1.** Let  $X : M \rightarrow \mathbb{R}^{n+1}$  be an  $n$ -dimensional compact  $\lambda$ -hypersurface in the Euclidean space  $\mathbb{R}^{n+1}$ . If

$$(1.9) \quad \left(H - \frac{\lambda}{2}\right)^2 \leq n + \frac{\lambda^2}{4},$$

then  $(H - \frac{\lambda}{2})^2 \equiv n + \frac{\lambda^2}{4}$  and  $M$  is isometric to the sphere  $S^n(r)$  with radius  $r > 0$ .

## 2. PROOFS OF THEOREMS FOR $\lambda$ -HYPERSURFACES

In order to prove our theorems, we prepare several fundamental formulas. Let  $X : M^n \rightarrow \mathbb{R}^{n+1}$  be an  $n$ -dimensional connected hypersurface of the  $(n+1)$ -dimensional Euclidean space  $\mathbb{R}^{n+1}$ . We choose a local orthonormal frame field  $\{e_A\}_{A=1}^{n+1}$  in  $\mathbb{R}^{n+1}$  with dual coframe field  $\{\omega_A\}_{A=1}^{n+1}$ , such that, restricted to  $M^n$ ,  $e_1, \dots, e_n$  are tangent to  $M^n$ . Then we have

$$dX = \sum_i \omega_i e_i, \quad de_i = \sum_j \omega_{ij} e_j + \omega_{in+1} e_{n+1}$$

and

$$de_{n+1} = \sum_i \omega_{n+1i} e_i.$$

We restrict these forms to  $M^n$ , then

$$\omega_{n+1} = 0, \quad \omega_{n+1i} = -\sum_{j=1}^n h_{ij} \omega_j, \quad h_{ij} = h_{ji},$$

where  $h_{ij}$  denotes components of the second fundamental form of  $X : M^n \rightarrow \mathbb{R}^{n+1}$ .  $H = \sum_{j=1}^n h_{jj}$  is the mean curvature and  $II = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$  is the second fundamental form of  $X : M^n \rightarrow \mathbb{R}^{n+1}$  with  $N = e_{n+1}$ . Let

$$h_{ijk} = \nabla_k h_{ij} \quad \text{and} \quad h_{ijkl} = \nabla_l \nabla_k h_{ij},$$

where  $\nabla_j$  is the covariant differentiation operator. Gauss equations, Codazzi equations and Ricci formulas are given by

$$(2.1) \quad R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk},$$

$$(2.2) \quad h_{ijk} = h_{ikj},$$

$$(2.3) \quad h_{ijkl} - h_{ijlk} = \sum_{m=1}^n h_{im} R_{mjkl} + \sum_{m=1}^n h_{mj} R_{mikl},$$

where  $R_{ijkl}$  is components of the curvature tensor. For a function  $F$ , we denote covariant derivatives of  $F$  by  $F_{,i} = \nabla_i F$ ,  $F_{,ij} = \nabla_j \nabla_i F$ . For  $\lambda$ -hypersurfaces, an elliptic operator  $\mathcal{L}$  is given by

$$(2.4) \quad \mathcal{L}f = \Delta f - \langle X, \nabla f \rangle,$$

where  $\Delta$  and  $\nabla$  denote the Laplacian and the gradient operator of the  $\lambda$ -hypersurface, respectively. The  $\mathcal{L}$  operator is introduced by Colding and Minicozzi in [6] for self-shrinkers and by Cheng and Wei [4] for  $\lambda$ -hypersurfaces.

The following lemma due to Colding and Minicozzi [6] is needed in order to prove our results.

**Lemma 2.1.** *Let  $X : M \rightarrow \mathbb{R}^{n+1}$  be a complete hypersurface. If  $u, v$  are  $C^2$  functions satisfying*

$$(2.5) \quad \int_M (|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|)e^{-\frac{|X|^2}{2}} d\mu < +\infty,$$

*then we have*

$$(2.6) \quad \int_M u(\mathcal{L}v)e^{-\frac{|X|^2}{2}} d\mu = - \int_M \langle \nabla u, \nabla v \rangle e^{-\frac{|X|^2}{2}} d\mu.$$

*Proof of theorem 1.1.* Since  $\langle X, N \rangle + H = \lambda$ , one has

$$(2.7) \quad H_{,i} = \sum_j h_{ij} \langle X, e_j \rangle,$$

$$H_{,ik} = \sum_j h_{ijk} \langle X, e_j \rangle + h_{ik} + \sum_j h_{ij} h_{jk} (\lambda - H).$$

From the Codazzi equation (2.2), we infer

$$\Delta H = \sum_i H_{,ii} = \sum_i H_{,i} \langle X, e_i \rangle + H + S(\lambda - H).$$

Hence, we get

$$(2.8) \quad \mathcal{L}H = \Delta H - \sum_i \langle X, e_i \rangle H_{,i} = H + S(\lambda - H),$$

$$(2.9) \quad \frac{1}{2} \mathcal{L}H^2 = |\nabla H|^2 + H^2 + S(\lambda - H)H.$$

By making use of the Ricci formulas and the Gauss equations and the Codazzi equations, we have

$$\begin{aligned} \mathcal{L}h_{ij} &= \Delta h_{ij} - \sum_k \langle X, e_k \rangle h_{ijk} \\ &= \sum_k h_{ijkk} - \sum_k \langle X, e_k \rangle h_{ijk} \\ &= (1 - S)h_{ij} + \lambda \sum_k h_{ik} h_{kj}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\frac{1}{2}\mathcal{L}S &= \frac{1}{2}\left\{\Delta\sum_{i,j}(h_{ij})^2 - \sum_k\langle X, e_k\rangle\left(\sum_{i,j}(h_{ij}^2)\right)_{,k}\right\} \\
&= \sum_{i,j,k}h_{ijk}^2 + (1-S)\sum_{i,j}h_{ij}^2 + \lambda\sum_{i,j,k}h_{ik}h_{kj}h_{ji} \\
&= \sum_{i,j,k}h_{ijk}^2 + (1-S)S + \lambda f_3,
\end{aligned}$$

where  $f_3 = \sum_{i,j,k}h_{ij}h_{jk}h_{ki}$ .

Taking  $\{e_1, e_2, \dots, e_n\}$  such that  $h_{ij} = \lambda_i\delta_{ij}$  at a point  $p$  and putting  $\mu_i = \lambda_i - \frac{H}{n}$ , we have

$$f_3 = \sum_i \lambda_i^3 = \sum_i \left(\mu_i + \frac{H}{n}\right)^3 = B_3 + \frac{3}{n}HB + \frac{1}{n^2}H^3$$

with  $B = \sum_i \mu_i^2 = S - \frac{H^2}{n}$  and  $B_3 = \sum_i \mu_i^3$ . Thus, we have

$$\begin{aligned}
\frac{1}{2}\mathcal{L}B &= \frac{1}{2}\mathcal{L}S - \frac{1}{2}\mathcal{L}\frac{H^2}{n} \\
&= \sum_{i,j,k}h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + (1-S)S + \lambda f_3 - \frac{H^2}{n} - S(\lambda - H)\frac{H}{n} \\
&= \sum_{i,j,k}h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + (1-B)B - \frac{1}{n}H^2B + \lambda B_3 + \frac{2}{n}\lambda HB.
\end{aligned}$$

Since

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = B,$$

it is not hard to prove

$$|B_3| \leq \frac{n-2}{\sqrt{n(n-1)}}B^{\frac{3}{2}}$$

and the equality holds if and only if at least,  $n-1$  of  $\mu_i$  are equal. Thus, we have

$$\begin{aligned}
\frac{1}{2}\mathcal{L}B &\geq \sum_{i,j,k}h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \\
&\quad + (1-B)B - \frac{1}{n}H^2B - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}B^{\frac{3}{2}} + \frac{2}{n}\lambda HB \\
&= \sum_{i,j,k}h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \\
&\quad + B\left(1 - B - \frac{1}{n}H^2 - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}B^{\frac{1}{2}} + \frac{2}{n}\lambda H\right) \\
&= \sum_{i,j,k}h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \\
&\quad + B\left(1 + \frac{n\lambda^2}{4(n-1)} - \frac{1}{n}(H-\lambda)^2 - (\sqrt{B} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}})^2\right).
\end{aligned}$$

Since  $X : M \rightarrow \mathbb{R}^{n+1}$  has polynomial area growth, according to the results of Cheng and Wei in [4], we can apply the lemma 2.1 to functions 1 and  $B = S - \frac{H^2}{n}$ . Hence, we have

$$\begin{aligned} 0 &\geq \int_M \left\{ \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 \right\} e^{-\frac{|X|^2}{2}} d\mu \\ &+ \int_M B \left( 1 + \frac{n\lambda^2}{4(n-1)} - \frac{1}{n} (H - \lambda)^2 - (\sqrt{B} + |\lambda| \frac{n-2}{2\sqrt{n(n-1)}})^2 \right) e^{-\frac{|X|^2}{2}} d\mu. \end{aligned}$$

From the Codazzi equations and the Schwarz inequality, we have

$$\begin{aligned} \sum_{i,j,k} h_{ijk}^2 &= 3 \sum_{i \neq k} h_{iik}^2 + \sum_i h_{iii}^2 + \sum_{i \neq j \neq k \neq i} h_{ijk}^2, \quad \frac{1}{n} |\nabla H|^2 \leq \sum_{i,k} h_{iik}^2, \\ \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 &\geq 2 \sum_{i \neq k} h_{iik}^2 + \sum_{i \neq j \neq k \neq i} h_{ijk}^2 \geq 0 \end{aligned}$$

and the equality holds if and only if  $h_{ijk} = 0$  for any  $i, j, k$ . Therefore, we get either  $B \equiv 0$  and  $X : M \rightarrow \mathbb{R}^{n+1}$  is totally umbilical; or there exists  $p \in M$  such that

$$(2.10) \quad \left( \sqrt{S(p) - \frac{H^2(p)}{n}} + |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n} (H(p) - \lambda)^2 > 1 + \frac{n\lambda^2}{4(n-1)};$$

or for any point of  $M$

$$\begin{aligned} \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 &= 0, \\ \left( \sqrt{S - \frac{H^2}{n}} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n} (H - \lambda)^2 &= 1 + \frac{n\lambda^2}{4(n-1)}. \end{aligned}$$

Hence, we know that the second fundamental form is parallel,  $X : M \rightarrow \mathbb{R}^{n+1}$  is an isoparametric complete hypersurface. If  $\lambda = 0$ , then  $X : M \rightarrow \mathbb{R}^{n+1}$  is isometric to the sphere  $S^n(\sqrt{n})$ , the Euclidean space  $\mathbb{R}^n$ , the cylinder  $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ . If  $\lambda \neq 0$ , then the number of the distinct principal curvatures is two and one of them is simple,  $X : M \rightarrow \mathbb{R}^{n+1}$  is isometric to the sphere  $S^n(r)$ , the Euclidean space  $\mathbb{R}^n$ , the cylinder  $S^1(r) \times \mathbb{R}^{n-1}$ , the cylinder  $S^{n-1}(r) \times \mathbb{R}$ . The proof of theorem 1.1 is completed.  $\square$

By making use of the same assertions as in Cheng and Peng [2], we know that the following generalized maximum principle holds.

**Theorem 2.1.** (*Generalized maximum principle for  $\mathcal{L}$ -operator*) Let  $X : M^n \rightarrow \mathbb{R}^{n+1}$  be a complete  $\lambda$ -hypersurface with Ricci curvature bounded from below. Let  $f$  be any  $C^2$ -function bounded from above on this  $\lambda$ -hypersurface. Then, there exists a sequence of points  $\{p_k\} \subset M$ , such that

$$(2.11) \quad \lim_{k \rightarrow \infty} f(X(p_k)) = \sup f, \quad \lim_{k \rightarrow \infty} |\nabla f|(X(p_k)) = 0, \quad \limsup_{k \rightarrow \infty} \mathcal{L}f(X(p_k)) \leq 0.$$

*Proof of Theorem 1.3.* From the proof in the theorem 1.1, we have

$$\begin{aligned} \frac{1}{2}\mathcal{L}B &\geq \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \\ &\quad + B\left(1 + \frac{n\lambda^2}{4(n-1)} - \frac{1}{n}(H-\lambda)^2 - (\sqrt{B} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}})^2\right) \end{aligned}$$

and

$$\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \geq 2 \sum_{i \neq k} h_{iik}^2 + \sum_{i \neq j \neq k \neq i} h_{ijk}^2 \geq 0.$$

Hence, we obtain

$$\frac{1}{2}\mathcal{L}B \geq B\left(1 + \frac{n\lambda^2}{4(n-1)} - \frac{1}{n}(H-\lambda)^2 - (\sqrt{B} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}})^2\right).$$

Since

$$\sup\left\{\left(\sqrt{S - \frac{H^2}{n}} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H-\lambda)^2\right\} < 1 + \frac{n\lambda^2}{4(n-1)},$$

we know  $H^2$  and  $S$  are bounded. Hence, from the Gauss equations, we infer that the Ricci curvature is bounded from below. Applying the generalized maximum principle for  $\mathcal{L}$  of  $\lambda$ -hypersurfaces to function  $B$ , there exists a sequence of points  $\{p_k\} \subset M$  such that

$$0 \geq \sup B\left(1 + \frac{n\lambda^2}{4(n-1)} - \sup\left\{\frac{1}{n}(H-\lambda)^2 + (\sqrt{B} + |\lambda|\frac{n-2}{2\sqrt{n(n-1)}})^2\right\}\right).$$

Hence,  $\sup B = 0$ , that is,  $S \equiv \frac{H^2}{n}$  and  $X : M \rightarrow \mathbb{R}^{n+1}$  is isometric to

- (1) the sphere  $S^n(r)$  with radius  $0 < r < \sqrt{n}$  or
- (2) the Euclidean space  $\mathbb{R}^n$ .

It completes the proof of the theorem 1.3. □

*Proof of Theorem 1.2.* By a direct calculation, one obtains

$$\begin{aligned} \frac{1}{2}\Delta|X|^2 &= \langle \Delta X, X \rangle + \sum_i \langle X_{,i}, X_{,i} \rangle \\ (2.12) \quad &= H \langle N, X \rangle + n \\ &= n + \frac{\lambda^2}{4} - (H - \frac{\lambda}{2})^2. \end{aligned}$$

Since the assumption of polynomial area growth, we have

$$\int_M (\Delta|X|^2) e^{-\frac{|X|^2}{2}} d\mu < +\infty, \quad \int_M |\nabla|X|^2|^2 e^{-\frac{|X|^2}{2}} d\mu < +\infty,$$

then we can apply the lemma 2.1 to function 1 and  $|X|^2$  and obtain

$$\frac{1}{4} \int_M |\nabla|X|^2|^2 e^{-\frac{|X|^2}{2}} d\mu = \frac{1}{2} \int_M (\Delta|X|^2) e^{-\frac{|X|^2}{2}} d\mu = \int_M \left(n + \frac{\lambda^2}{4} - (H - \frac{\lambda}{2})^2\right) e^{-\frac{|X|^2}{2}} d\mu.$$



From  $(H - \frac{\lambda}{2})^2 \geq n + \frac{\lambda^2}{4}$ , we get

$$(2.13) \quad \left(H - \frac{\lambda}{2}\right)^2 = n + \frac{\lambda^2}{4}, \quad \langle X, X \rangle = r^2,$$

namely,  $M$  is isometric to the sphere  $S^n(r)$  with radius  $r > 0$ . It completes the proof of the proposition 1.2.  $\square$

*Proof of Proposition 1.1.* Integrating (2.12) over  $M$  and using the Stokes formula, one concludes

$$(2.14) \quad \int_M \left(n + \frac{\lambda^2}{4} - (H - \frac{\lambda}{2})^2\right) d\mu = 0,$$

then it follows from  $(H - \frac{\lambda}{2})^2 \leq n + \frac{\lambda^2}{4}$  that

$$(2.15) \quad (H - \frac{\lambda}{2})^2 = n + \frac{\lambda^2}{4}$$

and  $M$  is isometric to the sphere  $S^n(r)$  with radius  $r > 0$ . It completes the proof of the proposition 1.1.  $\square$

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